4. PRINCIPAL AXES AND MOMENTS OF INERTIA

OF DEFORMABLE SYSTEMS

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SUMMARY

Information about principal axes and moments of inertia is presented in terms of formulas involving solely quantities which can be expressed in literal form whenever central principal axes can be located by inspection for at least one state of a deformable system composed of particles and rigid bodies. Two illustrative examples are worked out in detail.

INTRODUCTION

Principal axes and moments of inertia play important physical roles in certain situations. For example, any completely free rigid body (or a deformable body moving as if it were rigid) can execute a simple rotational motion, that is, a motion during which the angular velocity vector remains parallel to a body-fixed line; but this is possible only if the line is parallel to a central principal axis of inertia, and the stability of the motion is affected by the relative magnitudes of the central principal moments of inertia.

Principal axes and moments of inertia are of interest also from an analytical point of view, for their use can lead to marked simplifications of expressions for kinetic energy, angular momentum, gravity torque, etc. Consequently, the following are natural questions: Are there any difficulties associated with the use of principal axes and principal moments of inertia? And, if so, how can they be overcome? The answer to the first question is "yes"; for, while the problem of locating principal axes and evaluating principal moments of inertia can always be solved in principle (it is simply the eigenvalue problem for a 3×3 symmetric matrix), the solution, in general, entails finding the roots of a cubic equation, and this can give rise to difficulties ranging from relatively minor ones, presenting themselves when one is dealing with numerical (rather than literal) values of system parameters, to apparently insurmountable ones, which arise when one seeks results expressed entirely in literal form. As to the second question, it is the purpose of this paper to supply a partial answer by presenting formulas containing information about principal axes and moments of inertia in terms of quantities which

are readily available in literal form whenever central principal axes can be located by inspection for at least one state of the system under consideration. These formulas are

$$\tilde{b}_{jj} = 1 \quad , \quad \tilde{b}_{jk} = 0 \tag{1}$$

$$\tilde{b}_{jj,r} = 0 \quad , \quad \tilde{b}_{jk,r} = \frac{\tilde{I}_{jk,r}}{\tilde{I}_{jj} - \tilde{I}_{kk}} \tag{2}$$

$$\tilde{I}_j = \tilde{I}_{jj} \tag{3}$$

$$\tilde{I}_{j,r} = \tilde{I}_{jj,r} \tag{4}$$

$$\tilde{I}_{j,rs} = \tilde{I}_{jj,rs} + 2\left(\frac{\tilde{I}_{jk,r}\tilde{I}_{jk,s}}{\tilde{I}_{jj} - \tilde{I}_{kk}} + \frac{\tilde{I}_{j\ell,r}\tilde{I}_{j\ell,s}}{\tilde{I}_{jj} - \tilde{I}_{\ell\ell}}\right) \tag{5}$$

To explain the symbols appearing in Eqs. (1)-(5), it is helpful to refer to a schematic representation of the situation to which these equations apply, Fig. 1, where S designates a material system composed of particles and rigid bodies. The relative positions and orientations of the objects forming S are presumed to be governed by n scalar quantities $q_1, \ldots, q_n$ chosen in such a way that all vanish when S assumes a certain configuration called the reference state. S^* is the mass center of S. A_1, A_2, A_3 are mutually perpendicular axes intersecting at S^* and meeting two requirements: the orientation of each axis relative to S depends uniquely on the values of $q_1, \ldots, q_n$; and each axis is a principal axis of inertia of S for S^* when S is in the reference state. B_1, B_2, B_3 are instantaneous central principal axes of S; that is, they are principal axes of S for S^* for all values of $q_1, \ldots, q_n$. Finally, $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$, $\underset{\sim}{a}_3$ are unit vectors respectively parallel to A_1, A_2, A_3 and $\underset{\sim}{b}_1$, $\underset{\sim}{b}_2$, $\underset{\sim}{b}_3$ are unit vectors respectively parallel to B_1, B_2, B_3.

In Eqs. (1)-(5), each of the subscripts j, k, and ℓ may take on the values 1, 2, and 3, but no two may have the same value; the subscripts r and s assume the values $1, \ldots, n$; tildes denote evaluations at $q_1 = \ldots = q_n = 0$, that is, in the reference state; and the symbols appearing in the equations are defined as follows:

$$b_{jj} \triangleq \underset{\sim}{b}_j \cdot \underset{\sim}{a}_j \,, \quad b_{jk} \triangleq \underset{\sim}{b}_j \cdot \underset{\sim}{a}_k \tag{6}$$

$$I_{jj} \triangleq \underset{\sim}{a}_j \cdot \underset{\sim}{I} \cdot \underset{\sim}{a}_j \,, \quad I_{jk} \triangleq \underset{\sim}{a}_j \cdot \underset{\sim}{I} \cdot \underset{\sim}{a}_k \tag{7}$$

and

$$I_j \triangleq \underset{\sim}{b}_j \cdot \underset{\sim}{I} \cdot \underset{\sim}{b}_j \tag{8}$$

where $\underset{\sim}{I}$ is the inertia dyadic of S for S^*. Finally, a comma followed by r or/and s indicates partial differentiation with respect to r or/and s, so that, for example,

$$\tilde{I}_{11,57} = \left. \frac{\partial^2 I_{11}}{\partial q_5 \partial q_7} \right|_{q_1 = \cdots = q_n = 0}$$

EXAMPLES

One class of problems whose solution is facilitated by using Eqs. (1)-(4) involves questions regarding the sensitivity of principal axes orientations and principal moment of inertia values to small changes in the configuration of a deformable system. For example, consider the system S of three particles P, Q, and R shown in Fig. 2(a). If P and Q each have a mass m while R has a mass $2m$, the mass center S^* of S is situated as indicated, and X_1 and X_2 are central principal axes of S when the three particles form an equilateral triangle with sides of length $2L$. The associated moments of inertia have the values $2mL^2$ and $3mL^2$, respectively. In Fig. 2(b), S is shown in a state of distortion. The mass center S^* is again the midpoint of the line segment connecting R to O, the midpoint of P-Q, but lines passing through S^* and parallel to O-R or to P-Q are no longer central principal axes. Instead, two of the central principal axes of S are now the perpendicular lines B_1 and B_2, the first of which forms with O-R an angle θ that depends on the distortion, and the associated principal moments of inertia, I_1 and I_2, differ from $2mL^2$ and $3mL^2$. To study such distortion effects, one can introduce coordinates q_j, axes A_j, and unit vectors $\underset{\sim}{a}_j$ and $\underset{\sim}{b}_j$ $(j = 1,2,3)$ as shown in Fig. 2(b). [A_3, $\underset{\sim}{a}_3$, and $\underset{\sim}{b}_3$ are normal to the plane of the paper and are omitted from Fig. 2(b)]. The orientation of A_j relative to S then depends uniquely on the values of q_1, q_2, and q_3, and A_j is a principal axis of S for S^* when $q_1 = q_2 = q_3 = 0$.

Next, express $\underset{\sim}{b}_1$ as

$$\underset{\sim}{b}_1 \equiv \underset{\sim}{b}_1 \cdot \underset{\sim}{a}_1\underset{\sim}{a}_1 + \underset{\sim}{b}_1 \cdot \underset{\sim}{a}_2\underset{\sim}{a}_2$$

or, in accordance with Eqs. (6), as

$$\underset{\sim}{b}_1 = b_{11}\underset{\sim}{a}_1 + b_{12}\underset{\sim}{a}_2 \tag{9}$$

Then b_{11} and b_{12} are functions of q_1, q_2, and q_3. Expanding these in Taylor series, retaining only terms of degree lower than two, and using the summation convention for repeated subscripts, one can write

$$b_{11} \approx \tilde{b}_{11} + \tilde{b}_{11,r}q_r \ , \quad b_{12} \approx \tilde{b}_{12} + \tilde{b}_{12,r}q_r$$

or, after using Eqs. (1) and (2),

$$b_{11} \approx 1 \ , \ b_{12} \approx \frac{\tilde{I}_{12,r}}{\tilde{I}_{11} - \tilde{I}_{22}} q_r$$

Hence, from Eq. (9),

$$\underset{\sim}{b}_1 = \underset{\sim}{a}_1 + \frac{\tilde{I}_{12,r}}{\tilde{I}_{11} - \tilde{I}_{22}} q_r\underset{\sim}{a}_2 \tag{10}$$

Now, I_{11}, I_{22}, and I_{12} [see Eqs. (7)] can be formed readily since the A_1 and A_2 coordinates of P, Q, and R can be found by inspection:

$$I_{11} = 2m[(L + q_1)\cos q_3]^2 \tag{11}$$

$$I_{22} = m[3(L + q_2)^2 + 2(L + q_1)^2 \sin^2 q_3] \tag{12}$$

$$I_{12} = -2m(L + q_1)^2 \sin q_3 \cos q_3 \tag{13}$$

Setting $q_1 = q_2 = q_3 = 0$ in Eqs. (11) and (12), one obtains

$$\tilde{I}_{11} = 2mL^2 \ , \ \tilde{I}_{22} = 3mL^2$$

and partial differentiations of Eq. (13) yield

$$\tilde{I}_{12,1} = \tilde{I}_{12,2} = 0 \ , \ \tilde{I}_{12,3} = -2mL^2$$

Substituting into Eq. (10), one thus arrives at

$$\underset{\sim}{b}_1 = \underset{\sim}{a}_1 + 2q_3 \, \underset{\sim}{a}_2$$

which shows that during a sufficiently small distortion of S one can approximate θ [see Fig. 2(b)] with $2q_3$.

The principal moments of inertia I_1 and I_2 are also functions of q_1 , q_2 , and q_3 . Again resorting to series expansion, and using Eqs. (3) and (4), one can, therefore, write

$$I_1 \approx \tilde{I}_{11} + \tilde{I}_{11,r} q_r \ , \ I_2 \approx \tilde{I}_{22} + \tilde{I}_{22,r} q_r$$

and, in view of Eqs. (11)-(13),

$$I_1 \approx 2mL^2\left(1 + 2\,\frac{q_1}{L}\right) , \quad I_2 \approx 3mL^2\left(1 + 2\,\frac{q_2}{L}\right) \qquad (14)$$

These results describe the effect of a small distortion on I_1 and I_2 in terms of the quantities q_1 . q_2 and q_3 , which characterize the distortion.

Eqs. (14), and, indeed, the corresponding <u>exact</u> expressions for I_1 and I_2 , could be obtained also without the use of Eqs. (3) and (4), for we are here dealing with a planar distribution of matter, so that one needs to solve only a quadratic, rather than a cubic, equation to determine I_1 and I_2 . However, exact expressions are actually of less value than those displaying leading terms of series expansions when one is concerned with questions of sensitivity; and the reader can easily convince himself that the method here employed requires considerably less labor than does the process of finding exact expressions for I_1 and I_2 and then expanding in series.

A problem illustrating the use of Eqs. (4) and (5) to generate an <u>exact</u> result arises when one seeks conditions under which a principal moment of inertia of a deformable system possesses an extreme value. For instance, consider a system S composed of two rigid bodies, α and β , which are connected to each other by means of a gimbal γ , as shown in Fig. 3. Point 0 is the common mass center of α and β ; X_1 , X_2 , X_3 are principal axes of α , and Y_1 , Y_2 , Y_3 are principal axes of β ; and the gimbal can rotate relative to α and β only about X_1 and Y_2 , respectively. The relative orientation of α and β thus depends solely

on the angles q_1 and q_2, and the central principal axes of α and β are necessarily central principal axes of S when $q_1 = q_2 = 0$. Suppose now that B_3 is the central principal axis of S that coincides with X_3 and Y_3 when $q_1 = q_2 = 0$, and let I_3 be the associated central principal moment of inertia of S. Then I_3 has a (local) minimum value when $q_1 = q_2 = 0$ if the following conditions are satisfied:

$$\tilde{I}_{3,1} = \tilde{I}_{3,2} = 0 \tag{15}$$

$$\tilde{I}_{3,11} > 0 , \quad \tilde{I}_{3,11}\tilde{I}_{3,22} - (\tilde{I}_{3,12})^2 > 0 \tag{16}$$

How must α_1, α_2, α_3, the central principal moments of inertia of α, be related to β_1, β_2, β_3, the central principal moments of inertia of β, in order that Eqs. (15) and the inequalities (16) be satisfied? To answer this question, one can take for A_1, A_2, and A_3 the axes X_1, X_2, and X_3, in which case, from Eqs. (7),

$$\tilde{I}_{11} = \alpha_1 + \beta_1 , \quad \tilde{I}_{22} = \alpha_2 + \beta_2$$

$$I_{31} = (\beta_3 - \beta_1)c_1 s_2 c_2$$

$$I_{32} = -\beta_1 s_1 c_1 s_2^2 + \beta_2 s_1 c_1 - \beta_3 s_1 c_1 c_2^2$$

$$I_{33} = \alpha_3 + \beta_1 c_1^2 s_2^2 + \beta_2 s_1^2 + \beta_3 c_1^2 c_2^2$$

where s_i and c_i denote respectively $\sin q_i$ and $\cos q_i$ $(i=1,2)$. It follows that

$$\tilde{I}_{31,1} = 0 \qquad \tilde{I}_{32,1} = \beta_2 - \beta_3 \qquad \tilde{I}_{33,1} = 0$$

$$\tilde{I}_{31,2} = \beta_3 - \beta_1 \qquad \tilde{I}_{32,2} = 0 \qquad \tilde{I}_{33,2} = 0$$

$$\tilde{I}_{33,11} = 2(\beta_2 - \beta_3) \qquad \tilde{I}_{33,12} = 0 \qquad \tilde{I}_{33,22} = 2(\beta_1 - \beta_3)$$

and Eq. (4) thus gives

$$\tilde{I}_{3,1} = \tilde{I}_{3,2} = 0$$

while Eq. (5) yields

$$\tilde{I}_{3,11} = \tilde{I}_{33,11} + 2\left[\frac{(\tilde{I}_{31,1})^2}{\tilde{I}_{33} - \tilde{I}_{11}} + \frac{(\tilde{I}_{32,1})^2}{\tilde{I}_{33} - \tilde{I}_{22}}\right]$$

$$= \frac{2(\alpha_2 - \alpha_3)(\beta_2 - \beta_3)}{\alpha_2 - \alpha_3 + \beta_2 - \beta_3}$$

and

$$\tilde{I}_{3,12} = 0 \,, \quad \tilde{I}_{3,22} = \frac{2(\alpha_1 - \alpha_3)(\beta_1 - \beta_3)}{\alpha_1 - \alpha_3 + \beta_1 - \beta_3}$$

Eqs. (15) are thus seen to be satisfied automatically, and the inequalities (16) are equivalent to

$$\frac{(\alpha_2 - \alpha_3)(\beta_2 - \beta_3)}{\alpha_2 - \alpha_3 + \beta_2 - \beta_3} > 0$$

and

$$\frac{(\alpha_1 - \alpha_3)(\beta_1 - \beta_3)}{\alpha_1 - \alpha_3 + \beta_1 - \beta_3} > 0$$

Hence, when the central principal moments of inertia of α and β satisfy these two conditions, then I_3 has a (local) minimum at $q_1 = q_2 = 0$.

DERIVATIONS

To establish the validity of Eqs. (1)-(5), one may begin by observing that Eqs. (1) follow immediately from Eqs. (6) together with the fact that, by construction, $\underset{\sim}{a}_j = \underset{\sim}{b}_j$ when S is in the reference state. Next, use Eqs. (6) and the identity

$$\underset{\sim}{b}_1 \equiv \underset{\sim}{b}_1 \cdot \underset{\sim}{a}_1\underset{\sim}{a}_1 + \underset{\sim}{b}_1 \cdot \underset{\sim}{a}_2\underset{\sim}{a}_2 + \underset{\sim}{b}_1 \cdot \underset{\sim}{a}_3\underset{\sim}{a}_3$$

to write

$$\underset{\sim}{b}_1 = b_{11}\underset{\sim}{a}_1 + b_{12}\underset{\sim}{a}_2 + b_{13}\underset{\sim}{a}_3$$

and, after expanding b_{11}, b_{12}, and b_{13} in Taylor series and using Eqs. (1),

$$\underset{\sim}{b}_1 = (1 + \tilde{b}_{11,r}q_r + \dots)\underset{\sim}{a}_1 + (\tilde{b}_{12,r}q_r + \dots)\underset{\sim}{a}_2 + (\tilde{b}_{13,r}q_r + \dots)\underset{\sim}{a}_3 \tag{17}$$

Similarly. Eq. (3) is an immediate consequence of Eq. (8) and the first of Eqs. (7), and I_1 can, therefore, be expressed as

$$I_1 = \tilde{I}_{11} + \tilde{I}_{1,r}q_r + 2^{-1}\tilde{I}_{1,rs}q_r q_s + \dots \tag{18}$$

Now make use of the fact that, by construction, $\underset{\sim}{b}_1$ is parallel to a central principal axis of inertia of S for all values of $q_1, \dots, q_n$, so that

$$\underset{\sim}{I} \cdot \underset{\sim}{b}_1 = I_1\underset{\sim}{b}_1$$

or, after scalar multiplication of both sides of this equation with $\underset{\sim}{a}_1$,

$$\underset{\sim}{a}_1 \cdot \underset{\sim}{I} \cdot \underset{\sim}{b}_1 = I_1\underset{\sim}{a}_1 \cdot \underset{\sim}{b}_1 \tag{19}$$

Substitution for $\underset{\sim}{b}_1$ and I_1 from Eqs. (17) and (18), together with Eqs. (7), then gives

$$I_{11}\,(1 + \tilde{b}_{11,r}q_r + \dots) + I_{12}(\tilde{b}_{12,r}q_r + \dots) + I_{13}(\tilde{b}_{13,r}q_r + \dots)$$
$$= (\tilde{I}_{11} + \tilde{I}_{1,r}q_r + 2^{-1}I_{1,rs}q_r q_s + \dots)(1 + \tilde{b}_{11,r}q_r + \dots) \tag{20}$$

Moreover, I_{11}, I_{12}, and I_{13} can also be expanded in series:

$$I_{11} = \tilde{I}_{11} + \tilde{I}_{11,r}q_r + 2^{-1}\tilde{I}_{11,rs}q_r q_s + \dots$$

$$I_{12} = \tilde{I}_{12,r}q_r + \dots \quad , \quad I_{13} = \tilde{I}_{13,r}q_r + \dots$$

Consquently, each side of Eq. (20) can be regarded as a power series in $q_1, \dots, q_n$, and it follows that the coefficients of like terms can be equated separately. Doing this for terms of the first degree in $q_1, \dots, q_n$, one finds that

$$\tilde{I}_{1,r} = \tilde{I}_{11,r} \tag{21}$$

and considering second degree terms, one obtains

$$\tilde{I}_{1,rs} = \tilde{I}_{11,rs} + 2(\tilde{I}_{12,r}\tilde{b}_{12,s} + \tilde{I}_{13,r}\tilde{b}_{13,s}) \tag{22}$$

Proceeding similarly, but using $\underset{\sim}{a}_2$ and $\underset{\sim}{a}_3$ in place of $\underset{\sim}{a}_1$ in Eq. (19), one finds that

$$\tilde{b}_{12,r} = \frac{\tilde{I}_{12,r}}{\tilde{I}_{11} - \tilde{I}_{22}} \quad , \quad \tilde{b}_{13,r} = \frac{\tilde{I}_{13,r}}{\tilde{I}_{11} - \tilde{I}_{33}} \tag{23}$$

and substitution into Eq. (22) then leads to

$$\tilde{I}_{1,rs} = \tilde{I}_{11,rs} + 2\left(\frac{\tilde{I}_{12,r}\tilde{I}_{12,s}}{\tilde{I}_{11} - \tilde{I}_{22}} + \frac{\tilde{I}_{13,r}\tilde{I}_{13,s}}{\tilde{I}_{11} - \tilde{I}_{33}}\right) \tag{24}$$

Finally, since $\underset{\sim}{b}_1$ is a unit vector,

$$(b_{11})^2 + (b_{12})^2 + (b_{13})^2 = 1$$

Differentiation with respect to q_r gives

$$\tilde{b}_{11}\tilde{b}_{11,r} + \tilde{b}_{12}\tilde{b}_{12,r} + \tilde{b}_{13}\tilde{b}_{13,r} = 0$$

or, after using Eqs. (1),

$$\tilde{b}_{11,r} = 0 \tag{25}$$

For $j = 1$, the validity of Eqs. (2) is established by Eqs. (25) and (23), that of Eq. (4) by Eq. (21), and that of Eq. (5) by Eq. (24). Clearly, similar proofs can be carried out for $j = 2$ and $j = 3$.

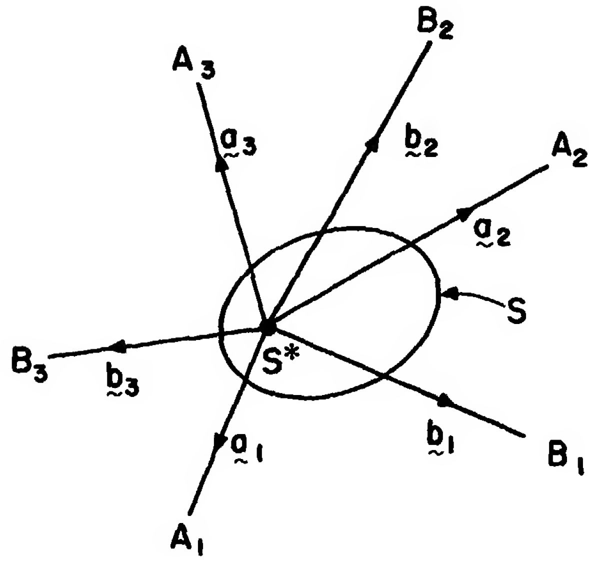
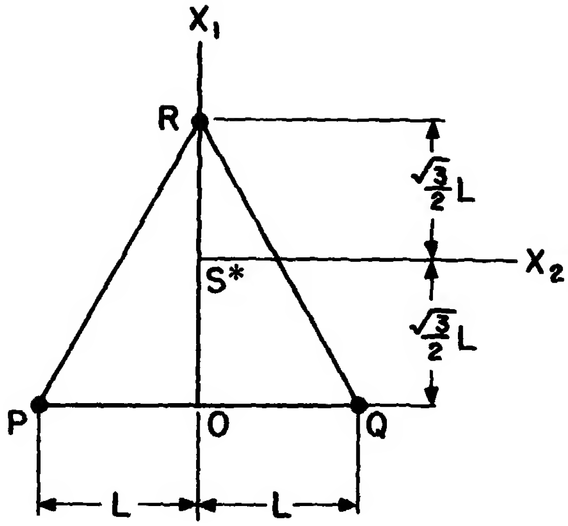
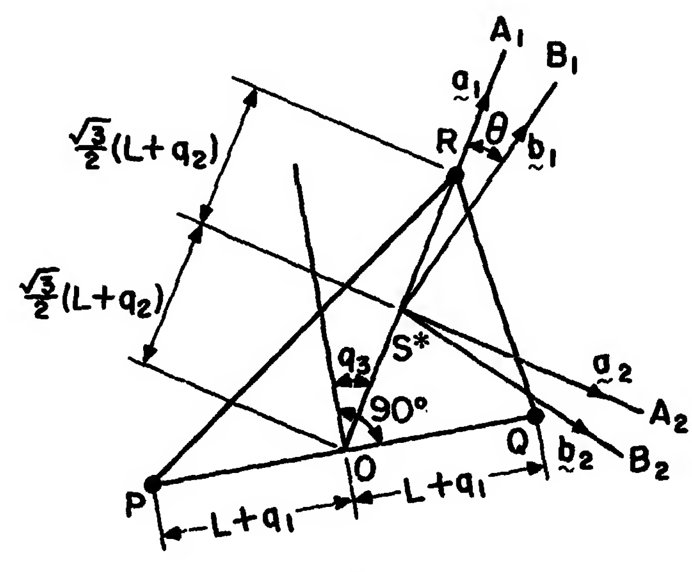


Figure 1.



X_1
R
$\frac{\sqrt{3}}{2}L$
X_2
S*
$\frac{\sqrt{3}}{2}L$
P
O
Q
L
L

(a)



A_1
B_1
$\underset{\sim}{a}_1$
θ
R
$\underset{\sim}{b}_1$
$\frac{\sqrt{3}}{2}(L+q_2)$
$\frac{\sqrt{3}}{2}(L+q_2)$
S*
q_3
90°
$\underset{\sim}{a}_2$
A_2
Q
$\underset{\sim}{b}_2$
B_2
O
P
$L+q_1$
$L+q_1$

(b)

Figure 2.

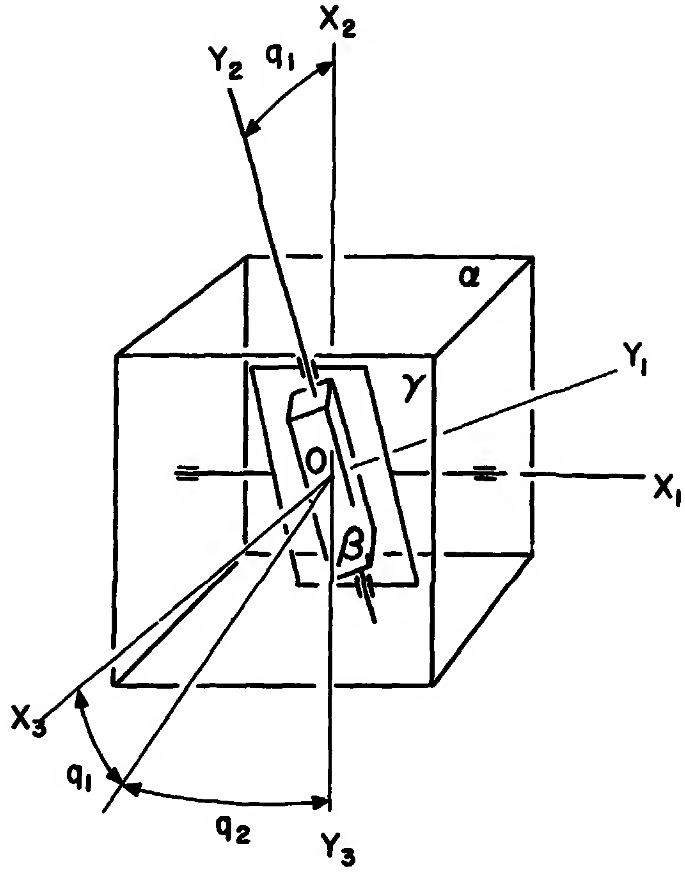


Figure 3.